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ORIGINAL PAPER

An efficient approach for solving second-order nonlinear differential equation with Neumann boundary conditions

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Abstract In this paper, a new straightforward approach based on a combination of Adomian decomposition method and Green's function for solving second-order Neumann boundary value problems is introduced. The proposed technique depends upon decomposing the domain of the problem into two sub-domains and constructing Green's function before establishing the recursive scheme for the solution components. The proposed method provides a direct recursive scheme for obtaining the series solution. Five illustrative examples are examined to demonstrate the accuracy, applicability, and generality of the proposed approach.

Keywords Neumann boundary value problems · Adomian decomposition method · Green's function · Approximations

1 Introduction

Many problems in science, technology, and engineering are formulated in boundary value problems, such as diffusion, heat flow problems, deflection in cables, chemical reaction, and heat and mass transfer within porous catalyst particle [1]. There are several types of boundary value problems (BVPs) depending on the boundary conditions usually given by Dirichlet boundary conditions, Neumann boundary conditions or mixed boundary conditions. The Neumann BC are usually the most physically

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reasonable choice. Such BVPs arise in chemical engineering, underground water flow and population dynamics, and other field of physics and mathematical chemistry [2,3]. Since it is usually impossible to obtain the closed-form solutions to BVPs met in practice, so these problems must be solved by various approximate and numerical methods.

The aim of this paper is to introduce an efficient approach for solving second-order differential equation with Neumann boundary conditions. Consider the following boundary value problems

$$y''(x) = f(x, y(x), y'(x)), \quad x \in [a, b], \tag{1.1}$$

subject to the Neumann boundary conditions

$$y'(a) = \alpha, \quad y'(b) = \beta, \tag{1.2}$$

where a and b are any finite real constants and f is continuous on the set $D = \{(x, y, y') : [a, b] \times \mathbb{R}^2\}$.

There are several analytical and numerical methods that are used in the literature to handle the BVPs [4–7]. The finite difference method [4], finite element method [8], the shooting method, collocation method [5], the variational iteration method [9,10] and many others are examples of these methods. These methods usually suffer from the huge size of calculations.

The ADM is a powerful technique for solving functional equations, linear and nonlinear. The ADM allows us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) without unphysical restrictive assumptions such as linearization, discretization, perturbation and guessing the initial term or a set of basis functions. Many authors [11–22] have shown interest to study of the ADM for different scientific models. The ADM provides a systematic approach for approximate analytic solutions of nonlinear and stochastic operator equations, including differential and integral equations.

It should be noted that in the literature very little attention has been devoted for applying the ADM to boundary value problems with Neumann boundary conditions. In this paper, we propose a practical method that can be efficiently used to handle Neumann BVPs. The method combines the Adomian method and the Green's function method to present a reliable technique that gives higher accuracy level. It also depends on decomposing the domain of the problem into two sub-domains as will be seen later.

1.1 Standard ADM

According to the standard ADM the problem (1.1) is written as

$$Ly(x) = Ny(x), \quad x \in [a, b], \tag{1.3}$$

where $L = \frac{d^2}{dx^2}$ and $Ny(x) = f(x, y(x), y'(x))$, and the inverse operator of $L^{-1}[\cdot] := \int_a^x \int_a^x [\cdot] dx dx$ is used to Eq. (1.3) resulting in

$$y(x) = c_1 + (x-a)c_2 + L^{-1}Ny(x), \tag{1.4}$$

where c_1 and c_2 are unknown parameter to be determined. The idea of ADM is based on decomposing $y(x)$ and the nonlinear term $Ny(x)$ by an infinite series as

$$y(x) = \sum_{j=0}^{\infty} y_j(x), \quad Ny(x) = \sum_{j=0}^{\infty} A_j, \tag{1.5}$$

where A_j are Adomian's polynomials which can be obtained by using the formula given in [17] as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} y_k\lambda^k\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{1.6}$$

Substituting (1.5) in (1.4), yields the following recursive scheme

$$y_0 = c_1 + (x-a)c_2, \quad y_j = L^{-1}[A_{j-1}], \quad j = 1, 2 \ldots. \tag{1.7}$$

The approximate solution is defined as $\phi_n(x, c_1, c_2) = \sum_{j=0}^{n} y_j(x)$. Note that the approximate solution ϕ_n depends on the unknown parameters c_1 and c_2. To evaluate these parameters, we begin with $\phi_n = c_1 + (x-a)c_2 + L^{-1}[\sum_{j=1}^{n-1} A_{j-1}]$. Imposing the boundary conditions (1.2), we have $\phi_n'(a) := c_2 + \left[L^{-1}[\sum_{j=1}^{n-1} A_{j-1}]\right]_{x=a}' = \alpha$, $\phi_n'(b) := c_2 + \left[L^{-1}[\sum_{j=1}^{n-1} A_{j-1}]\right]_{x=b}' = \beta$. Note that by solving these with respect to c_2, its value can be obtained, while the parameter c_1 remains unknown. Hence we can not proceed with the standard ADM. In the next section, we introduce a modification of the ADM which combines with Green's function to overcome the difficulties occurring in the standard ADM for solving BVPs (1.1) and (1.2).

2 Two-stage ADM

In order to solve BVPs (1.1) and (1.2) by the ADM, we divide original problem into two subproblems. To do so, we first decompose the domain of the problem $[a, b]$ into two sub-domains $[a, c] \cup [c, b]$, set $y(c) = \eta$, $c \in (a, b)$, where η is an unknown constant.

For $[a, c]$: We consider the following boundary value problem

$$y''(x) = f(x, y(x), y'(x)), \quad y'(a) = \alpha, \quad y(c) = \eta. \tag{2.1}$$

Integrating the problem (2.1) w.r.t. x from a to x and using $y'(a) = \alpha$, we obtain the Volterra integro-differential equation

$$y'(x) = \alpha + \int_a^x f(\xi, y(\xi), y'(\xi))d\xi. \tag{2.2}$$

Again integrating above equation w.r.t. x from a to x and changing the order of integration we get the Volterra integral equation

$$y(x) = y(a) + \alpha(x-a) + \int_a^x (x-\xi) f(\xi, y(\xi), y'(\xi)) d\xi. \tag{2.3}$$

Using other boundary condition $y(c) = \eta$, the unknown constant $y(a)$ is identified as

$$y(a) = \eta - \alpha(c-a) - \int_a^c (c-\xi) f(\xi, y(\xi), y'(\xi)) d\xi.$$

Substituting $y(a)$ in Eq. (2.3), we obtain the Fredholm–Volterra integral equation

$$y(x) = \eta + \alpha(x-c) - \int_a^c (c-\xi) f(\xi, y(\xi), y'(\xi)) d\xi + \int_a^x (x-\xi) f(\xi, y(\xi), y'(\xi)) d\xi.$$

Splitting the first integral into two parts from a to x and x to c, we get

$$y(x) = \eta + \alpha(x-c) + \int_a^x (\xi - c) f(\xi, y(\xi), y'(\xi)) d\xi + \int_x^c (\xi - c) f(\xi, y(\xi), y'(\xi)) d\xi + \int_a^x (x-\xi) f(\xi, y(\xi), y'(\xi)) d\xi.$$

Combining the first and last integrals, we have

$$y(x) = \eta + \alpha(x-c) + \int_a^x (x-c) f(\xi, y(\xi), y'(\xi)) d\xi + \int_x^c (\xi - c) f(\xi, y(\xi), y'(\xi)) d\xi.$$

Thus we obtain the Fredholm integral equation as

$$y(x) = \eta + \alpha(x - c) + \int_a^c G(x, \xi) f(\xi, y(\xi), y'(\xi)) d\xi, \tag{2.4}$$

where the Green's function $G(x, \xi)$ is

$$G(x, \xi) = \begin{cases} (\xi - c), & a \le x \le \xi, \\ (x - c), & \xi \le x \le c. \end{cases} \tag{2.5}$$

Substituting (1.5) into (2.4), we obtain

$$\sum_{j=0}^{\infty} y_j(x) = \eta + \alpha(x - c) + \int_a^c G(x, \xi) \left[\sum_{j=0}^{\infty} A_j \right] d\xi. \tag{2.6}$$

Comparing both sides of (2.6), we have

$$\left.\begin{aligned} y_0(x, \eta) &= \eta + \alpha(x - c), \\ y_j(x, \eta) &= \int_a^c G(x, \xi) A_{j-1} d\xi, \quad j = 1, 2 \ldots \end{aligned}\right\} \tag{2.7}$$

and the modified recursive scheme is defined as

$$\left.\begin{aligned} y_0(x, \eta) &= \eta, \\ y_1(x, \eta) &= \alpha(x - c) + \int_a^c G(x, \xi) A_0 d\xi, \\ y_j(x, \eta) &= \int_a^c G(x, \xi) A_{j-1} d\xi, \quad j = 2, 3 \ldots \end{aligned}\right\} \tag{2.8}$$

which leads to a complete determination of the components $y_j(x)$, and the n-terms series solution of the sub-problem (2.1) is given by

$$\psi_n^{(I)}(x) = \sum_{j=0}^{n} y_j(x, \eta). \tag{2.9}$$

For $[c, b]$: Let us consider the following boundary value problem

$$y''(x) = f(\xi, y(x), y'(x)), \quad y(c) = \eta, \quad y'(b) = \beta, \tag{2.10}$$

Integrating (2.10) w.r.t x from c to x, we get

$$y'(x) = y'(c) + \int_c^x f(\xi, y(\xi), y'(\xi)) d\xi. \tag{2.11}$$

Integrating (2.11) from c to x and using boundary condition $y(c) = \eta$ and changing the order of integration we have

$$y(x) = \eta + y'(c)(x - c) + \int_c^x (x - \xi) f(\xi, y(\xi), y'(\xi)) d\xi. \tag{2.12}$$

To eliminate the unknown constant $y'(c)$ from (2.12), we impose other boundary condition $y'(b) = \beta$ in equation (2.11) and yields

$$y'(c) = \beta - \int_c^b f(\xi, y(\xi), y'(\xi)) d\xi. \tag{2.13}$$

Combining equations (2.12) and (2.13), we get

$$\begin{aligned} y(x) &= \eta + \beta(x - c) + \int_c^b (c - x) f(\xi, y(\xi), y'(\xi)) d\xi \\ &\quad + \int_c^x (x - \xi) f(\xi, y(\xi), y'(\xi)) d\xi \\ &= \eta + \beta(x - c) + \int_c^x (c - \xi) f(\xi, y(\xi), y'(\xi)) d\xi \\ &\quad + \int_x^b (c - x) f(\xi, y(\xi), y'(\xi)) d\xi. \end{aligned}$$

Hence, we have the Fredholm integral equation as

$$y(x) = \eta + \beta(x - c) + \int_c^b G(x, \xi) f(\xi, y(\xi), y'(\xi)) d\xi, \tag{2.14}$$

where $G(x, \xi)$ is given by

$$G(x, \xi) = \begin{cases} (c - x), & c \le x \le \xi, \\ (c - \xi), & \xi \le x \le b. \end{cases} \tag{2.15}$$

By substituting (1.5) into (2.14), we obtain

$$\sum_{j=0}^{\infty} y_j(x) = \eta + \beta(x - c) + \int_c^b G(x, \xi) \left[\sum_{j=0}^{\infty} A_j \right] d\xi. \tag{2.16}$$

Upon comparing both sides of (2.16), we have the following recursion scheme

$$\left.\begin{aligned} &y_0(x,\eta) = \eta + \beta(x-c), \\ &y_j(x,\eta) = \int_c^b G(x,\xi)A_{j-1}d\xi, \quad j = 1, 2\ldots \end{aligned}\right\} \tag{2.17}$$

and the modified recursive scheme is defined as

$$\left.\begin{aligned} &y_0(x) = \eta, \\ &y_1(x) = \beta(x-c) + \int_c^b G(x,\xi)A_0 d\xi, \\ &y_j(x) = \int_c^b G(x,\xi)A_{j-1}d\xi, \quad j = 2, 3\ldots \end{aligned}\right] \tag{2.18}$$

which gives a complete determination of the components $y_j(x)$, we denote the n-terms approximant of the series solutions

$$\psi_n^{(II)}(x) = \sum_{j=0}^{n} y_j(x,\eta). \tag{2.19}$$

In order to determine η form equations (2.11) and (2.19), we use the continuity condition for the flux

$$\left.\frac{d\psi_n^{(I)}(x,\eta)}{dx}\right|_{x=c} - \left.\frac{d\psi_n^{(II)}(x,\eta)}{dx}\right|_{x=c} = 0, \quad n = 1, 2, \ldots \tag{2.20}$$

which leads to a sequence of equations. By solving these equations, we can obtain approximate value of η. Then the approximate series solution of original BVPs (1.1) and (1.2) is defined as

$$\psi_n(x) = \begin{cases} \psi_n^{(I)}(x,\eta_n) := \sum_{j=0}^{n} y_j(x,\eta_n), & a \le x < c, \\ \psi_n^{(II)}(x,\eta_n) := \sum_{j=0}^{n} y_j(x,\eta_n), & c \le x \le b, \end{cases} \tag{2.21}$$

where $\eta_n, n = 1, 2, 3, \ldots$ are approximate values of η.

3 Convergence analysis

Note that the convergence of the ADM for differential and integral equations have already been discussed in [12,23]. In this section we follow the approach discussed [12] for the convergence of the recursive schemes (2.7) or (2.8) and (2.17) or (2.18).

For $[a, c]$: Let $\mathbb{E} = C^1[a, c]$ be a Banach space with a norm defined by

$$\|y\|_\infty = \max_{x\in[a,c]} (l_1|y(x)| + l_2|y'(x)|), \quad y \in \mathbb{E}, \tag{3.1}$$

where l_1 and l_2 are Lipschitz constants defined by (3.4). Rewrite integral equation (2.4) in an operator form

$$y(x) = \mathcal{T} y(x), \tag{3.2}$$

where $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is defined as

$$\mathcal{T} y(x) = \eta + \alpha(x - c) + \int_a^c G(x, \xi) f(\xi, y(\xi), y'(\xi))d\xi. \tag{3.3}$$

Theorem 3.1 *Let* $\mathbb{E}$ *be Banach space with norm given by* (3.1)*. Assume that* $f(x, y, y')$ *satisfies Lipschitz condition, i.e., there exists constants* l_1 *and* l_2 *such that for all* (x, y, y')*,* $(x, z, z') \in D$*,*

$$|f(x, y, y') - f(x, z, z')| \le l_1|y - z| + l_2|(y' - z')|. \tag{3.4}$$

If $\delta := (l_1 m_1 + l_1 m_2) < 1$*, then the Eq.* (3.2) *has a unique solution in* $\mathbb{E}$*.*

Proof For any $y, y^* \in \mathbb{E}$, we have

$$\begin{aligned} |\mathcal{T} y(x) - \mathcal{T} y^*(x)| &= \left| \int_a^c G(x, \xi) \left[f(\xi, y(\xi), y'(\xi)) - f(\xi, y^*(\xi), y'^*(\xi)) \right] d\xi \right| \\ &\le \int_a^c |G(x, \xi)| \left| f(\xi, y(\xi), y'(\xi)) - f(\xi, y'(\xi), y'^*(\xi)) \right| d\xi \\ &\le m_1 \max_{x\in[a,c]} \left[l_1|y(\xi) - y(\xi)| + l_2|y'(\xi) - y'^*(\xi)| \right] \\ &= m_1\|y - y^*\|_\infty, \end{aligned} \tag{3.5}$$

where $m_1 := \int_a^c |G(x, \xi)|d\xi$. Similarly, consider

$$\begin{aligned} \left| \frac{d}{dx} \left(\mathcal{T} y(x) - \mathcal{T} y^*(x) \right) \right| &= \left| \int_a^c G_x(x, \xi) \left[f(\xi, y(\xi), y'(\xi)) - f(\xi, y^*(\xi), y'^*(\xi)) \right] d\xi \right| \\ &\le \int_a^c |G_x(x, \xi)| \left| f(\xi, y(\xi), y'(\xi)) - f(\xi, y'(\xi), y'^*(\xi)) \right| d\xi \end{aligned}$$

$$\leq m_2 \max_{x\in[a,c]} (l_1|y(\xi) - y(\xi)| + l_2|y'(\xi) - y'^*(\xi)|)$$
$$= m_2\|y - y^*\|_\infty, \tag{3.6}$$

where $m_2 := \int_a^c |G_x(x, \xi)|d\xi$. Combining the estimates (3.5) and (3.6), we obtain

$$\|\mathcal{T}y - \mathcal{T}y^*\|_\infty \leq (l_1 m_1 + l_1 m_2)\|y - y^*\|_\infty = \delta\|y - y^*\|_\infty, \tag{3.7}$$

where $\delta = (l_1 m_1 + l_1 m_2)$. If $\delta < 1$, then $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is contraction mapping and hence by the Banach contraction mapping theorem, the Eq. (3.2) has a unique solution in $\mathbb{E}$. □

Theorem 3.2 *Assume that all the conditions of Theorem 3.1 hold. Let $y_0, y_1, y_2, \ldots,$ be the solution components obtained by the recursive schemes (2.7) or (2.8), and let $\psi_n = \sum_{j=0}^{n} y_j$ be the n-terms series solution defined by (2.9). Then ψ_n converges to the exact solution y of the operator Eq. (3.2) whenever $\delta < 1$ and $\|y_1\| < \infty$.*

Proof Using (2.7) or (2.8) and (2.9), we have

$$\psi_n = y_0 + \sum_{j=1}^{n} y_j = \eta + \alpha(x - c) + \sum_{j=1}^{n}\left[\int_0^1 G(x, \xi)A_{j-1}d\xi\right]$$
$$= \eta + \alpha(x - c) + \int_0^1 G(x, \xi)q(\xi)\sum_{j=0}^{n-1} A_j d\xi. \tag{3.8}$$

For all $n, m \in \mathbb{N}$, with $n > m$, consider

$$\|\psi_n - \psi_m\| = \max_{a\leq x\leq c}\left|\int_a^c G(x, \xi)\left[\sum_{j=0}^{n-1} A_j - \sum_{j=0}^{m-1} A_j\right]d\xi\right|. \tag{3.9}$$

Using the relation $\sum_{j=0}^{n-1} A_j \leq f(x, \psi_{n-1}, \psi'_{n-1})$ in above Eq. ([24] pp. 945), we obtain

$$\|\psi_n - \psi_m\| \leq \max_{0\leq x\leq 1}\left|\int_a^c G(x, \xi)(f(\xi, \psi_{n-1}, \psi'_{n-1}) - f(\xi, \psi_{m-1}, \psi'_{m-1}))d\xi\right|. \tag{3.10}$$

Hence for any $n \in \mathbb{N}$ and following the steps of Theorem 3.1, we can obtain the following relation

$$\|\psi_{n+1} - \psi_n\| \leq \delta\|\psi_n - \psi_{n-1}\|.$$

Thus we have

$$\|\psi_{n+1}-\psi_n\| \le \delta\|\psi_n-\psi_{n-1}\| \le \delta^2\|\psi_{n-1}-\psi_{n-2}\| \le \ldots \le \delta^n\|\psi_1-\psi_0\|.$$

For any $n, m \in \mathbb{N}$, with $n > m$, consider

$$\begin{aligned}\|\psi_n-\psi_m\| &\le \|\psi_n-\psi_{n-1}\| + \|\psi_{n-1}-\psi_{n-2}\| + \cdots + \|\psi_{m+1}-\psi_m\| \\ &\le [\delta^{n-1}+\delta^{n-2}+\cdots+\delta^m]\|\psi_1-\psi_0\| \\ &= \delta^m[1+\delta+\delta^2+\cdots+\delta^{n-m-1}]\|\psi_1-\psi_0\| \\ &= \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\|y_1\|.\end{aligned}$$

Since $\delta < 1$ so, $(1-\delta^{n-m}) < 1$ and $\|y_1\| < \infty$, it follows that

$$\|\psi_n-\psi_m\| \le \frac{\delta^m}{1-\delta}\|y_1\| \to 0, \quad \text{as} \quad m \to \infty.$$

This implies that there exits a ψ such that $\lim_{n\to\infty}\psi_n = \psi$. Since, we have $y = \sum_{j=0}^{\infty} y_j = \lim_{n\to\infty}\psi_n$, that is, $y = \psi$ which is the exact solution of (3.2). □

For $[c, b]$: Let $\mathbb{E} = C^1[c, b]$ be a Banach space with a norm defined by

$$\|y\|_\infty = \max_{x\in[c,b]}(l_1|y(x)| + l_2|y'(x)|), \quad y \in \mathbb{E}. \tag{3.11}$$

Rewriting integral Eq. (2.14) in an operator form

$$y(x) = \mathcal{T}y(x), \tag{3.12}$$

where $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is defined as

$$\mathcal{T}y(x) = \eta + \beta(x-c) + \int_c^b G(x,\xi)f(\xi, y(\xi), y'(\xi))d\xi, \tag{3.13}$$

Note 3.1 *As we discussed the convergence analysis for the recursive schemes* (2.7) *or* (2.8)*, by following similar steps we can also do for the recursive schemes* (2.17) *or* (2.18).

4 Numerical results

We consider five illustrative examples to demonstrate the accuracy and efficiency of the ADM. All symbolic and numerical computations are performed by using 'Mathematica' 8.0 software package. Numerical results obtained by the proposed method are compared with the exact and known results.

Example 4.1 Consider the following second-order BVPs with Neumann boundary conditions [25]

$$\left.\begin{array}{l} y''(x) = -(y(x) + x) \qquad x \in [0, 1], \\ y'(0) = -1 + \operatorname{cosec}(1), \quad y'(1) = -1 + \cot(1), \end{array}\right\} \tag{4.1}$$

where the exact solution is $y(x) = -x + \operatorname{cosec}(1)\sin(x)$.

To get approximate solution of above example, we apply two-stage ADM with Green's function. To apply the proposed method, we first decompose the domain of solution [0, 1] into two sub-domains [0, 0.5] and [0.5, 1]. Suppose $y(0.5) = \eta$, where η is an unknown constant. According to the proposed method, we solve the following two sub-BVPs

$$y''(x) = -(y(x) + x), \quad y'(0) = -1 + \operatorname{cosec}(1), \quad y(0.5) = \eta, \quad x \in [0, 0.5], \tag{4.2}$$

and

$$y''(x) = -(y(x) + x), \quad y(0.5) = \eta, \quad y'(1) = -1 + \cot(1), \quad x \in [0.5, 1]. \tag{4.3}$$

For $[0, 0.5]$: According to the recursive scheme (2.8) with $a = 0$, $c = 0.5$ and $\alpha = -1 + \operatorname{cosec}(1)$, we transform sub-problem (4.2) into the following recursive scheme as

$$\left.\begin{array}{l} y_0(x) = \eta, \\ y_1(x) = (-1 + \operatorname{cosec}(1))(x - 0.5) + \displaystyle\int_0^{0.5} G(x, \xi)(-y_0(\xi) - \xi)d\xi, \\ y_j(x) = \displaystyle\int_0^{0.5} G(x, \xi)(-y_{j-1}(\xi))d\xi, \quad j = 2, 3, \ldots \end{array}\right\} \tag{4.4}$$

where the Green's function $G(x, \xi)$ is given by

$$G(x, \xi) = \begin{cases} (\xi - 0.5), \ 0 \le x \le \xi, \\ (x - 0.5), \ \xi \le x \le 0.5. \end{cases} \tag{4.5}$$

Using (4.4) and (4.5), we obtain the solution components

$$\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= \frac{25}{48} - x - \frac{x^3}{6} + \eta\left(\frac{1}{8} - \frac{x^2}{2}\right) - \operatorname{cosec}(1)\left(\frac{1}{2} + x\right), \\ y_2(x) &= \frac{169}{3840} - \frac{25x^2}{96} + \frac{x^3}{6} + \frac{x^5}{120} + \eta\left(\frac{5}{384} - \frac{x^2}{16} + \frac{x^4}{24}\right) \end{aligned}$$

$$- \operatorname{cosec}(1)\left(\frac{1}{24}+\frac{x^2}{4}-\frac{x^3}{6}\right),$$
$$\vdots$$

Hence, the n-terms series solution can obtained as $\psi_n^{(I)}(x,\eta)=\sum_{j=0}^{n} y_j$.

For [0.5, 1]: According to the recursive scheme (2.18) with $c = 0.5$, $b = 1$ and $\beta = -1 + \cot(1)$, we convert sub-problem (4.3) into the following recursive scheme

$$\left.\begin{aligned} y_0(x) &= \eta,\\ y_1(x) &= (-1+\cot(1))(x-0.5)+\int_{0.5}^{1} G(x,\xi)(-y_0(\xi)-\xi)d\xi,\\ y_j(x) &= \int_{0.5}^{1} G(x,\xi)(-y_{j-1}(\xi))d\xi, \quad j=2,3,\ldots \end{aligned}\right\} \tag{4.6}$$

where $G(x, \xi)$ is given by

$$G(x,\xi)=\begin{cases}(0.5-x),\ 0.5\le x\le \xi,\\ (0.5-\xi),\ \xi\le x\le 1.\end{cases} \tag{4.7}$$

By using (4.6) and (4.7), we obtain the solution components

$$\begin{aligned} y_0(x) &= \eta,\\ y_1(x) &= \frac{13}{48}-\frac{x}{2}-\frac{x^3}{6}-\eta\left(\frac{3}{8}-x+\frac{x^2}{2}\right)-\cot(1)\left(\frac{1}{2}-x\right),\\ y_2(x) &= \frac{43}{1280}-\frac{x}{48}-\frac{13x^2}{96}+\frac{x^3}{12}+\frac{x^5}{120}-\eta\left(\frac{1}{128}+\frac{x}{24}-\frac{3x^2}{16}+\frac{x^3}{6}-\frac{x^4}{24}\right)\\ &\quad -\cot(1)\left(\frac{1}{24}-\frac{1}{4}x^2+\frac{1}{6}x^3\right),\\ &\ \,\vdots \end{aligned}$$

Hence, the n-terms series solution can obtained as $\psi_n^{(II)}(x,\eta)=\sum_{j=0}^{n} y_j$. To determine unknown constant η, we use the continuity condition for the flux defined as (2.20)

$$\left.\frac{d\psi_n^{(I)}(x,\eta)}{dx}\right|_{x=0.5}-\left.\frac{d\psi_n^{(II)}(x,\eta)}{dx}\right|_{x=0.5}=0, \quad n=1,2,\ldots \tag{4.8}$$

Solving above equation numerically, we obtain a sequence of approximate values for η. The numerical values of η are listed in Table 1. From Table 1, we observe that the approximate value of η approaches the value 0.0697469.... Note that the actual value of η is $y(\frac{1}{2}) = -\frac{1}{2} + \operatorname{cosec}(1)\sin(\frac{1}{2}) = 0.06974696....$

Table 1 Approximate value of η for $n = 2, 3, 4 \ldots, 8$.

n	2	3	4	5	6	7	8
η_n	0.0673141	0.0694995	0.0697219	0.0697444	0.0697467	0.0697469	0.0697470

Table 2 Maximum absolute error of Example 4.1

n	2	3	4	5	6	7	8
E_n	2.446E−03	2.486E−04	2.522E−05	2.556E−06	2.589E−07	2.624E−08	2.656E−09

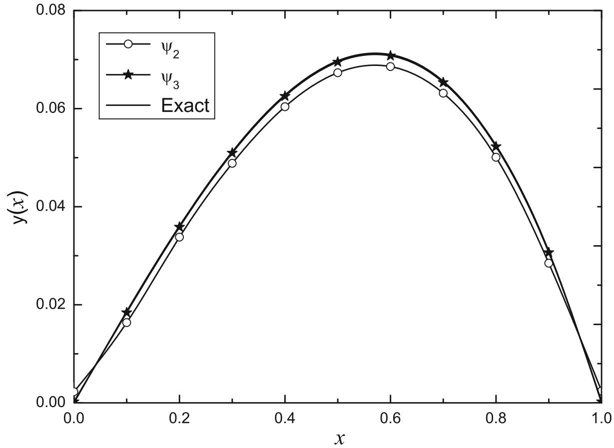

Fig. 1 Exact y and approximate $\psi_n, n = 2, 3$ of Example 4.1

In view of Eq. (2.20), we can obtain approximate series solution of the problem (4.1) as

$$\psi_n(x) = \begin{cases} \psi_n^{(I)}(x, \eta_n), & 0 \leq x < 0.5, \\ \psi_n^{(II)}(x, \eta_n), & 0.5 \leq x \leq 1, \end{cases} \tag{4.9}$$

where η_n is an approximate value of η. In order to show the accuracy and efficiency of the proposed method, we define maximum absolute error as $E_n = |\psi_n(x) - y(x)|, \quad n = 1, 2, \ldots,$ where $y(x)$ is the exact solution and $\psi_n(x)$ is n-terms series solution defined by equation (2.21). The maximum absolute error $E_n, n = 2, 3, \ldots 8$ are listed in Table 2. Moreover, we plot the exact $y(x)$ and the approximate solutions ψ_2, ψ_3 in Figure 1, where ψ_3 and the exact solution overlap each others.

Example 4.2 Consider the following second-order nonlinear BVPs with Neumann boundary conditions

$$\left.\begin{aligned} &y''(x) = -e^{-2y(x)}, \quad x \in [0, 1], \\ &y'(0) = 1, \quad y'(1) = \frac{1}{2}, \end{aligned}\right\} \tag{4.10}$$

where the exact solution is $y(x) = \ln(1 + x)$.

As we did in the last example, the domain of solution [0, 1] is decomposed into two sub-domains [0, 0.5] and [0.5, 1] and setting $y(0.5) = \eta$. We solve the following two nonlinear sub-BVPs

$$y''(x) = -e^{-2y(x)}, \quad y'(0) = 1, \quad y(0.5) = \eta, \quad x \in [0, 0.5], \tag{4.11}$$

and

$$y''(x) = -e^{-2y(x)}, \quad y(0.5) = \eta, \quad y'(1) = \frac{1}{2}, \quad x \in [0.5, 1]. \tag{4.12}$$

For $[0, 0.5]$: According to the recursive scheme (2.8) with $a = 0$, $c = 0.5$, and $\alpha = 1$, we transform the sub-problem (4.11) into the following recursive scheme as

$$\left.\begin{aligned} &y_0(x) = \eta, \\ &y_1(x) = 1(x - 0.5) + \int_0^{0.5} G(x, \xi) A_0 d\xi, \\ &y_j(x) = \int_0^{0.5} G(x, \xi) A_{j-1} d\xi, \quad j = 2, 3, \ldots \end{aligned}\right\} \tag{4.13}$$

where $G(x, \xi)$ is same as in (4.5). We use the Duan's algorithm [26] for obtaining the Adomian's polynomial for $f(y) = -e^{-2y}$ given as

$$A_0 = -e^{-2y_0}, \quad A_1 = 2e^{-2y_0} y_1, \quad A_2 = 2e^{-2y_0}(-y_1^2 + y_2), \ldots \tag{4.14}$$

Using (4.13) and (4.14), the solution components are computed as

$$\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= \frac{1}{8}\left(-4 + 8x + e^{-2\eta}\left(1 - 4x^2\right)\right), \\ y_2(x) &= \frac{1}{192} e^{-4\eta}\left(-5 + 24x^2 - 16x^4 + 16e^{2\eta}\left(1 - 6x^2 + 4x^3\right)\right), \\ &\vdots \end{aligned}$$

Thus, the n-terms series solution is obtained as $\psi_n^{(I)}(x, \eta) = \sum_{j=0}^{n} y_j$.

For $[0.5, 1]$: According to the recursive scheme (2.18) with $c = 0.5$, $b = 1$ and $\beta = \frac{1}{2}$, we convert sub-problem (4.12) into the following recursive scheme as

Table 3 Approximate value of η for $n = 2, 3, 4 \ldots, 8$.

n	2	3	4	5	6	7	8
η_n	0.3687141	0.4134221	0.4049372	0.4063283	0.4050268	0.4055677	0.4054158

Table 4 Maximum absolute error of Example 4.2

n	2	3	4	5	6	7	8
E_n	4.052E−02	9.233E−03	1.696E−03	9.613E−04	5.123E−04	1.389E−04	6.432E−05

$$\left.\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= \tfrac{1}{2}(x-0.5) + \int_{0.5}^{1} G(x,\xi)A_0 d\xi, \\ y_j(x) &= \int_{0.5}^{1} G(x,\xi)A_{j-1} d\xi, \quad j = 2, 3, \ldots \end{aligned}\right\} \tag{4.15}$$

where $G(x, \xi)$ is same as in (4.7). By using (4.15) and (4.14), the solution components are computed as

$$\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= \frac{1}{8}e^{-2\eta}\left(3 + 2e^{2\eta} - 2x\right)(-1+2x), \\ y_2(x) &= -\frac{1}{192}e^{-4\eta}(-1+2x)\left(3 + 22x - 28x^2 + 8x^3 + e^{2\eta}\left(8 + 16x - 16x^2\right)\right), \\ &\vdots \end{aligned}$$

Thus, the n-terms series solution is obtained as $\psi_n^{(II)}(x, \eta) = \sum_{j=0}^{n} y_j$. In order to determine value of η, we again use the continuity condition for the flux defined by (2.20), i.e.,

$$\left.\frac{d\psi_n^{(I)}(x,\eta)}{dx}\right|_{x=0.5} - \left.\frac{d\psi_n^{(II)}(x,\eta)}{dx}\right|_{x=0.5} = 0, \quad n = 1, 2, \ldots \tag{4.16}$$

After solving above Eq. (4.16) numerically, we get a sequence of approximate values for η. From Table 3 we can observe that the approximate value of η approaches to 0.4054158.... Note that the actual value of η is $y(\frac{1}{2}) = \ln(1.5) = 0.4054651081081\ldots$. Table 4 shows the numerical results of the maximum absolute error E_n, $n = 2, 3, \ldots 8$. Further, we plot the exact $y(x)$ and the approximate solutions ψ_n for $n = 2, 3, 4$ in Fig. 2. It can be observed from the figure that ψ_4 and the exact solution overlap.

Example 4.3 Consider the nonlinear oscillator second-order nonlinear BVPs with Neumann boundary conditions [25]

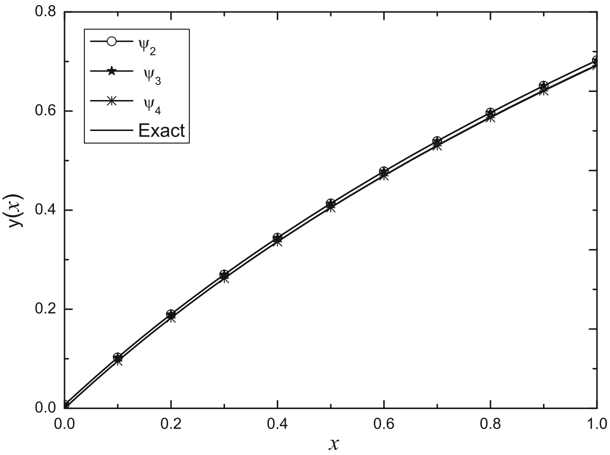

Fig. 2 Exact y and approximate ψ_n, $n = 2, 3$ of Example 4.2

$$\left.\begin{aligned} &y'' + \omega^2 y = \lambda y^m, \quad x \in [0, 1], \\ &y'(0) = 1, \quad y'(1) = cn\left(1|\frac{1}{4}\right) dn\left(1|\frac{1}{4}\right), \end{aligned}\right\} \tag{4.17}$$

where m is a positive integer. This problem has the exact solution $y = sn(x|\frac{1}{4})$ when $m = 3$ (Duffing oscillator), $\lambda = \frac{1}{2}$ and $\omega^2 = \frac{5}{4}$, where sn, cn, dn are Jacobi elliptic functions.

As we did before, we solve the following two nonlinear sub-BVPs

$$y''(x) = \lambda y^m(x) - \omega^2 y(x), \quad y'(0) = 1, \quad y(0.5) = \eta, \quad x \in [0, 0.5], \tag{4.18}$$

and

$$y''(x) = \lambda y^m(x) - \omega^2 y(x), \quad y(0.5) = \eta, \quad y'(1) = cn\left(1|\frac{1}{4}\right) dn\left(1|\frac{1}{4}\right), \quad x \in [0.5, 1]. \tag{4.19}$$

For $[0, 0.5]$: Using the recursive scheme (2.7) with $a = 0$, $c = 0.5$ and $\alpha = 1$, we convert sub-problem (4.18) into the following recursive scheme

$$\left.\begin{aligned} &y_0(x) = \eta + 1(x - 0.5), \\ &y_j(x) = \int_0^{0.5} G(x, \xi)(\lambda A_{j-1} - \omega^2 y_{j-1}) d\xi, \quad j = 1, 2, \ldots \end{aligned}\right\} \tag{4.20}$$

where $G(x, \xi)$ is defied as in (4.5). The first few terms of Adomian polynomials for $f(y) = y^m$ is obtained as

$$A_0 = y_0^m, \quad A_1 = m y_0^{m-1} y_1, \quad A_2 = m y_0^{m-1} y_2 + \frac{1}{2} m(m-1) y_0^{m-2} y_1^2, \ldots \tag{4.21}$$

In view of (4.20) and (4.21), the solution components are obtained as

$$\begin{aligned}
y_0(x) &= \eta + (x - 0.5),\\
y_1(x) &= -\frac{47}{960} + \frac{9x^2}{32} - \frac{7x^3}{48} - \frac{x^4}{16} + \frac{x^5}{40} + \frac{17\eta}{128} - \frac{7x^2\eta}{16} - \frac{x^3\eta}{4} + \frac{x^4\eta}{8} + \frac{\eta^2}{16}\\
&\quad - \frac{3x^2\eta^2}{8} + \frac{x^3\eta^2}{4} - \frac{\eta^3}{16} + \frac{x^2\eta^3}{4},\\
&\ \vdots
\end{aligned}$$

Hence, the n-terms series solution is obtained as $\psi_n^{(I)}(x, \eta) = \sum_{j=0}^{n} y_j$.

For [0.5, 1]: Applying the recursive scheme (2.17) with $c = 0.5$, $b = 1$ and $\beta = cn\left(1|\frac{1}{4}\right) dn\left(1|\frac{1}{4}\right)$, the sub-problem (4.19) is converted into the following recursive scheme

$$\left.\begin{aligned}
y_0(x) &= \eta + \left(cn\left(1|\tfrac{1}{4}\right) dn\left(1|\tfrac{1}{4}\right)\right)(x - 0.5),\\
y_j(x) &= \int_{0.5}^{1} G(x, \xi)(\lambda A_{j-1} - \omega^2 y_{j-1}) d\xi, \quad j = 1, 2, \ldots
\end{aligned}\right\} \tag{4.22}$$

where $G(x, \xi)$ is same as in (4.7). Using (4.22) and (4.21), the solution components are computed as

$$\begin{aligned}
y_0(x) &= \eta + \left(cn\left(1|\frac{1}{4}\right) dn\left(1|\frac{1}{4}\right)\right)(x - 0.5),\\
y_1(x) &= -0.026557 + 0.157602x^2 - 0.099268x^3 - 0.008699x^4 + 0.003479x^5\\
&\quad - 0.458259\eta + 1.21643x\eta - 0.574641x^2\eta - 0.0671448x^3\eta\\
&\quad + 0.0335724x^4\eta + 0.0323904\eta^2 - 0.194342x^2\eta^2 + 0.129562x^3\eta^2\\
&\quad + 0.1875\eta^3 - 0.5x\eta^3 + 0.25x^2\eta^3,\\
&\ \vdots
\end{aligned}$$

Thus, the n-terms series solution is obtained as $\psi_n^{(II)}(x, \eta) = \sum_{j=0}^{n} y_j$. To determine the approximate values η_n for η, we apply the continuity condition for the flux, i.e.

$$\left.\frac{d\psi_n^{(I)}(x, \eta)}{dx}\right|_{x=0.5} - \left.\frac{d\psi_n^{(II)}(x, \eta)}{dx}\right|_{x=0.5} = 0, \quad n = 1, 2, \ldots \tag{4.23}$$

Table 5 Approximate value of η for $n = 2, 3, 4 \ldots, 8$.

n	1	2	3	4	5	6	7
η_n	0.5055120	0.4752197	0.4748384	0.4750737	0.4750902	0.4750845	0.4750830

Table 6 Maximum absolute error of Example 4.3

n	1	2	3	4	5	6	7
E_n	3.158E−02	6.033E−04	3.327E−04	1.585E−05	8.018E−06	1.813E−06	7.454E−08

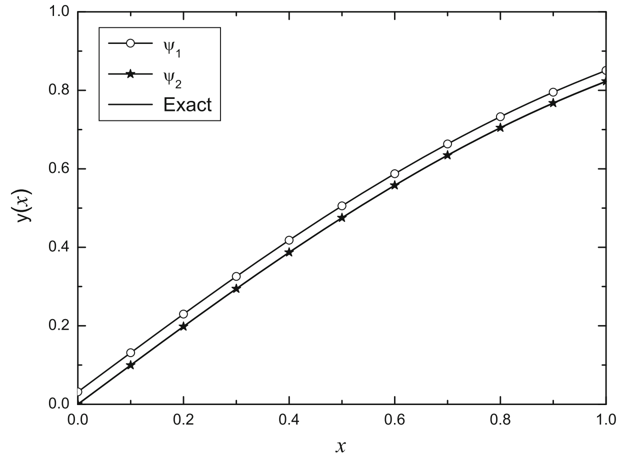

Fig. 3 Exact y and approximate $\psi_n, n = 1, 2$ of Example 4.3

By solving Eq. (4.23), we obtain the approximate values for η. From Table 5 we can observe that the approximate value of η approaches to 0.4750830.... Note that the actual value of η is $y(\frac{1}{2}) = sn(\frac{1}{2}|\frac{1}{4}) = 0.47508293602....$ In Table 6, we list the numerical results of the maximum absolute error $E_n, n = 1, 3, \ldots 7$. We also plot the exact $y(x)$ and the approximate solutions ψ_n for $n = 1, 2$ in Fig. 3. It can be seen from the figure that ψ_2 and the exact solution overlap.

Example 4.4 Consider the nonlinear second-order nonlinear BVPs with Neumann conditions

$$\left.\begin{aligned} &y''(x) = 4x^2 e^{2y(x)} - 2e^{y(x)}, \quad x \in [0, 1],\\ &y'(0) = 0, \quad y'(1) = \frac{-2}{5}. \end{aligned}\right\} \tag{4.24}$$

Its exact solution is $y(x) = \ln\left(\frac{1}{4+x^2}\right)$.

We decompose the domain of solution [0, 1] into two sub-domains [0, 0.5] and [0.5, 1], and set $y(0.5) = \eta$, where η is an unknown constant. Then we solve the following two sub-BVPs

$$y''(x) = 4x^2 e^{2y(x)} - 2e^{y(x)}, \quad y'(0) = 0, \quad y(0.5) = \eta, \quad x \in [0, 0.5], \tag{4.25}$$

and

$$y''(x) = 4x^2 e^{2y(x)} - 2e^{y(x)}, \quad y(0.5) = \eta, \quad y'(1) = \frac{-2}{5}, \quad x \in [0.5, 1]. \tag{4.26}$$

For [0, 0.5]: By using the recursive scheme (2.8) with $a = 0$, $c = 0.5$ and $\alpha = 0$, the sub-problem (4.25) is transformed into the following recursive scheme

$$\left.\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= 0(x - 0.5) + \int_0^{0.5} G(x, \xi) A_0 d\xi, \\ y_j(x) &= \int_0^{0.5} G(x, \xi) A_{j-1} d\xi, \quad j = 1, 2, \ldots \end{aligned}\right\} \tag{4.27}$$

where $G(x, \xi)$ is given by (4.5). The first few terms of Adomian polynomials for $f(x, y) = 4x^2 e^{2y} - 2e^y$ are obtained as

$$\begin{aligned} A_0 &= e^{y_0}(4x^2 - 2), \quad A_1 = 2e^{y_0}\left(4e^{y_0}x^2 - 1\right) y_1, \quad A_2 \\ &= e^{y_0}\left(8e^{y_0}x^2\left(y_1^2 + y_2\right) - y_1^2 - 2y_2\right), \ldots \end{aligned} \tag{4.28}$$

By using (4.27) and (4.28), the solution components are computed as

$$\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= \frac{e^\eta}{4} - \frac{e^{2\eta}}{48} - e^\eta x^2 + \frac{1}{3} e^{2\eta} x^4, \\ y_2(x) &= \frac{5e^{2\eta}}{96} - \frac{e^{3\eta}}{90} + \frac{11e^{4\eta}}{16128} - \frac{e^{2\eta}x^2}{4} + \frac{e^{3\eta}x^2}{48} + \frac{e^{2\eta}x^4}{6} + \frac{e^{3\eta}x^4}{6} \\ &\quad - \frac{e^{4\eta}x^4}{72} - \frac{13e^{3\eta}x^6}{45} + \frac{e^{4\eta}x^8}{21}, \\ &\vdots \end{aligned}$$

Hence, the n-terms series solution is obtained as $\psi_n^{(I)}(x, \eta) = \sum_{j=0}^{n} y_j$.

For [0.5, 1]: Applying the recursive scheme (2.18) with $c = 0.5$, $b = 1$ and $\beta = \frac{-2}{5}$, we convert sub-problem (4.19) into the following recursive scheme

$$\left.\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= (\tfrac{-2}{5})(x - 0.5) + \int_{0.5}^{1} G(x, \xi) A_0 d\xi, \\ y_j(x) &= \int_{0.5}^{1} G(x, \xi) A_{j-1} d\xi, \quad j = 1, 2, \ldots \end{aligned}\right\} \tag{4.29}$$

Table 7 Approximate value of η for $n = 2, 3, 4 \ldots, 8$.

n	1	2	3	4	5	6	7
η_n	−1.4369459	−1.4464431	−1.4466153	−1.4469142	−1.4469162	−1.4469190	−1.4469189

Table 8 Maximum absolute error of Example 4.4

n	1	2	3	4	5	6	7
E_n	7.538E−03	4.211E−04	1.983E−04	4.028E−06	1.954E−06	4.075E−08	2.662E−08

In view of (4.29) and (4.28), the solution components are computed as

$$\begin{aligned}
y_0(x) &= \eta,\\
y_1(x) &= \frac{1}{5} - \frac{3e^{\eta}}{4} + \frac{31e^{2\eta}}{48} - \frac{2x}{5} + 2e^{\eta}x - \frac{4}{3}e^{2\eta}x - e^{\eta}x^2 + \frac{1}{3}e^{2\eta}x^4,\\
y_2(x) &= \frac{e^{\eta}}{30} - \frac{403e^{2\eta}}{2400} + \frac{13e^{3\eta}}{48} - \frac{23561e^{4\eta}}{80640} + \frac{1}{10}e^{2\eta}x - \frac{37}{120}e^{3\eta}x + \frac{71}{126}e^{4\eta}x - \frac{e^{\eta}x^2}{5}\\
&\quad + \frac{3}{4}e^{2\eta}x^2 - \frac{31}{48}e^{3\eta}x^2 + \frac{2e^{\eta}x^3}{15} - \frac{2}{3}e^{2\eta}x^3 + \frac{4}{9}e^{3\eta}x^3 + \frac{3}{10}e^{2\eta}x^4 - \frac{1}{2}e^{3\eta}x^4\\
&\quad + \frac{31}{72}e^{4\eta}x^4 - \frac{4}{25}e^{2\eta}x^5 + \frac{4}{5}e^{3\eta}x^5 - \frac{8}{15}e^{4\eta}x^5 - \frac{13}{45}e^{3\eta}x^6 + \frac{1}{21}e^{4\eta}x^8,\\
&\vdots
\end{aligned}$$

Thus, the n-terms series solution is obtained as $\psi_n^{(II)}(x,\eta) = \sum_{j=0}^{n} y_j$. To evaluate the approximate values of η, we use the continuity condition for the flux, i.e.

$$\left.\frac{d\psi_n^{(I)}(x,\eta)}{dx}\right|_{x=0.5} - \left.\frac{d\psi_n^{(II)}(x,\eta)}{dx}\right|_{x=0.5} = 0, \quad n = 1, 2, \ldots \tag{4.30}$$

Solving Eq. (4.30), we obtain approximate values for η. From Table 7 we can observe that the approximate value of η approaches to $-1.4469189\ldots$. Also, it can be noted that the true value of η is $y(\frac{1}{2}) = \ln\left(\frac{4}{17}\right) = -1.4469189829363254\ldots$. Table 8 shows the numerical results of the maximum absolute error E_n, $n = 1, 2, \ldots 7$. Moreover, we plot the exact $y(x)$ and the approximate solutions ψ_n for $n = 1, 2$ in Fig. 4. It can be observed from the figure that ψ_2 and the exact solution overlap.

Example 4.5 Consider the nonlinear second-order nonlinear BVPs with Neumann conditions

$$\left.\begin{aligned}
&y''(x) - y'(x) = 9x^4e^{2y(x)} + 3xe^{y(x)}(x-2), \quad x \in [0, 1],\\
&y'(0) = 0, \quad y'(1) = \frac{-3}{7}.
\end{aligned}\right\} \tag{4.31}$$

This problem has the exact solution $y(x) = \ln\left(\frac{1}{6+x^3}\right)$.

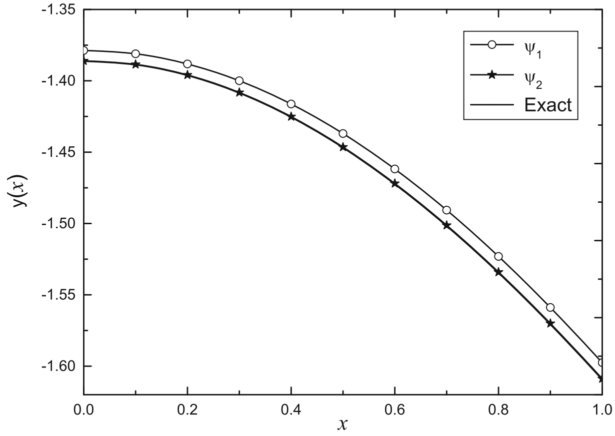

Fig. 4 Exact y and approximate $\psi_n, n = 1, 2$ solutions of Example 4.4

According to the proposed method we decompose the domain of solution [0, 1] into two sub-domains [0, 0.5] and [0.5, 1], and setting $y(0.5) = \eta$, where η is an unknown constant. We then solve the following two non-linear sub-BVPs

$$y''(x) - y'(x) = 9x^4 e^{2y(x)} + 3xe^{y(x)}(x-2), \quad y'(0) = 0, \quad y(0.5) = \eta, \quad x \in [0, 0.5], \tag{4.32}$$

and

$$y''(x) - y'(x) = 9x^4 e^{2y(x)} + 3xe^{y(x)}(x-2), \quad y(0.5) = \eta, \quad y'(1) = \frac{-3}{7}, \quad x \in [0.5, 1]. \tag{4.33}$$

For [0, 0.5]: According to the recursive scheme (2.8) with $a = 0$, $c = 0.5$ and $\alpha = 0$, the sub-problem (4.32) is transformed into the following recursive scheme

$$\left.\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= 0(x-0.5) + \int_0^{0.5} G(x,\xi)(y_0'(\xi) + A_0)d\xi, \\ y_j(x) &= \int_0^{0.5} G(x,\xi)(y_{j-1}'(\xi) + A_{j-1})d\xi, \quad j = 1, 2, \ldots \end{aligned}\right\} \tag{4.34}$$

where $G(x, \xi)$ is same as in (4.5). Similarly, using the Duan's efficient algorithm [26], the Adomian's polynomials for $f(x, y) = 9x^4 e^{2y} + 3xe^y(x-2)$ are obtained as

$$A_0 = 3e^{y_0}x\left(x - 2 + 3e^{y_0}x^3\right), \quad A_1 = 3e^{y_0}x\left(x - 2 + 6e^{y_0}x^3\right)y_1,$$
$$A_2 = \frac{3}{2}e^{y_0}x\left(\left(x - 2 + 12e^{y_0}x^3\right)y_1^2 + 2\left(x - 2 + 6e^{y_0}x^3\right)y_2\right), \ldots \tag{4.35}$$

Table 9 Approximate value of η for $n = 2, 3, 4 \ldots, 8$.

n	1	2	3	4	5	6	7
η_n	−1.2380495	−1.9637606	−1.7856098	−1.8154752	−1.8117846	−1.8125546	−1.8123149

Using (4.34) and (4.35), we obtain the solution components as

$$y_0(x) = \eta,$$
$$y_1(x) = \frac{7e^{\eta}}{64} - \frac{3e^{2\eta}}{640} - e^{\eta}x^3 + \frac{e^{\eta}x^4}{4} + \frac{3}{10}e^{2\eta}x^6,$$
$$\vdots$$

Hence, the n-terms series solution is obtained as $\psi_n^{(I)}(x, \eta) = \sum_{j=0}^{n} y_j$.

For [0.5, 1]: Using the recursive scheme (2.18) with $c = 0.5$, $b = 1$ and $\beta = \frac{-3}{7}$, we convert sub-problem (4.33) into the following recursive scheme

$$\left.\begin{aligned} y_0(x) &= \eta, \\ y_1(x) &= (\tfrac{-2}{5})(x - 0.5) + \int_{0.5}^{1} G(x, \xi)(y_0'(\xi) + A_0)d\xi, \\ y_j(x) &= \int_{0.5}^{1} G(x, \xi)(y_{j-1}'(\xi) + A_{j-1})d\xi, \quad j = 1, 2, \ldots \end{aligned}\right\} \tag{4.36}$$

where $G(x, \xi)$ is given by (4.7). By using (4.36) and (4.35), we have the solution components as

$$y_0(x) = \eta,$$
$$y_1(x) = \frac{3}{14} - \frac{57e^{\eta}}{64} + \frac{573e^{2\eta}}{640} - \frac{3x}{7} + 2e^{\eta}x - \frac{9}{5}e^{2\eta}x - e^{\eta}x^3 + \frac{e^{\eta}x^4}{4} + \frac{3}{10}e^{2\eta}x^6,$$
$$\vdots$$

Thus, the n-terms series solution is obtained as $\psi_n^{(II)}(x, \eta) = \sum_{j=0}^{n} y_j$. In order to determine value of η, we use the continuity condition for the flux, i.e.,

$$\left.\frac{d\psi_n^{(I)}(x, \eta)}{dx}\right|_{x=0.5} - \left.\frac{d\psi_n^{(II)}(x, \eta)}{dx}\right|_{x=0.5} = 0, \quad n = 1, 2, \ldots \tag{4.37}$$

Then solving Eq. (4.37), the approximate values for η are obtained. From Table 9 we can observe that the approximate value of η approaches to $-1.8123149\ldots$. It is noted that the actual value of η is $y(\frac{1}{2}) = \ln\left(\frac{8}{49}\right) = -1.81237875643079\ldots$. In Table 10 we list the numerical results of the maximum absolute error $E_n, n = 1, 2, \ldots 7$. Furthermore, we plot the exact $y(x)$ and the approximate solutions ψ_n for $n = 1, 2$ in

Table 10 Maximum absolute error of Example 4.5

n	1	2	3	4	5	6	7
E_n	3.265E−01	8.627E−02	1.523E−02	1.766E−03	3.382E−04	9.953E−05	3.637E−05

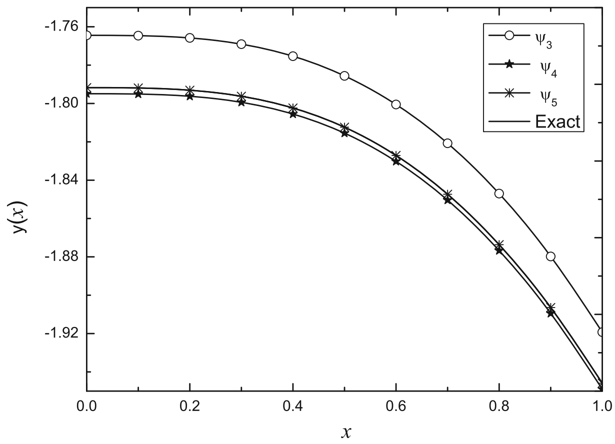

Fig. 5 Exact y and approximate ψ_n, $n = 3, 4, 5$ of Example 4.5

Figure 5. It can be seen from the figure that the approximate solution ψ_5 converges to the exact solution.

5 Conclusion

We have presented the ADM and Green's function for solving nonlinear second-order BVPs with Neumann boundary conditions. The proposed technique depends upon decomposing the domain of the problem into two sub-domains and constructing Green's function before establishing the recursive scheme for the solution components. Accuracy and efficiency of the ADM have been examined by solving five examples of second-order Neumann BVPs. Unlike the finite difference, the cubic spline methods, and any other discretization methods, the proposed method does not require any linearization or discretization of variables. Convergence analysis of the proposed method has also been discussed. The combination of ADM and Green functions show enhancement over existing techniques where we overcome the cumbersome work.

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References

1. R. Rach, J. Duan, A. Wazwaz, Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. J. Math. Chem. **52**(1), 255–267 (2014)

2. S. Khuri, A. Wazwaz, A variational approach for a class of nonlocal elliptic boundary value problems. J. Math. Chem. **52**(5), 1324–1337 (2014)
3. G. Rahamathunissa, L. Rajendran, Application of He's variational iteration method in nonlinear boundary value problems in enzyme-substrate reaction diffusion processes: part 1. The steady-state amperometric response. J. Math. Chem. **44**(3), 849–861 (2008)
4. M. Chawla, A fourth-order tridiagonal finite difference method for general non-linear two-point boundary value problems with mixed boundary conditions. IMA J. Appl. Math. **21**(1), 83–93 (1978)
5. I. Aziz, B. Šarler et al., The numerical solution of second-order boundary-value problems by collocation method with the haar wavelets. Math. Comput. Model. **52**(9), 1577–1590 (2010)
6. B. Jang, Two-point boundary value problems by the extended Adomian decomposition method. J. Comput. Appl. Math. **219**(1), 253–262 (2008)
7. R. Singh, J. Kumar, G. Nelakanti, Numerical solution of singular boundary value problems using Green's function and improved decomposition method. J. Appl. Math. Comput. **43**(1–2), 409–425 (2013)
8. I. Christie, D.F. Griffiths, A.R. Mitchell, O.C. Zienkiewicz, Finite element methods for second order differential equations with significant first derivatives. Int. J. Numer. Meth. Eng. **10**(6), 1389–1396 (1976)
9. A. Wazwaz, The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models. Commun. Nonlinear Sci. Numer. Simul. **16**(10), 3881–3886 (2011)
10. S. Momani, S. Abuasad, Z. Odibat, Variational iteration method for solving nonlinear boundary value problems. Appl. Math. Comput. **183**(2), 1351–1358 (2006)
11. L. Bougoffa, R.C. Rach, Solving nonlocal boundary value problems for first-and second-order differential equations by the Adomian decomposition method. Kybernetes **42**(4), 641–664 (2013)
12. R. Singh, J. Kumar, G. Nelakanti, Approximate series solution of fourth-order boundary value problems using decomposition method with Green's function. J. Math. Chem. **52**(4), 1099–1118 (2014)
13. J. Duan, An efficient algorithm for the multivariable Adomian polynomials. Appl. Math. Comput. **217**(6), 2456–2467 (2010)
14. J. Duan, New recurrence algorithms for the nonclassic Adomian polynomials. Comput. Math. Appl. **62**(8), 2961–2977 (2011)
15. S. Khuri, A. Sayfy, A novel approach for the solution of a class of singular boundary value problems arising in physiology. Math. Comput. Model. **52**(3), 626–636 (2010)
16. A. Wazwaz, A reliable study for extensions of the Bratu problem with boundary conditions. Math. Methods Appl. Sci. **35**(7), 845–856 (2012)
17. G. Adomian, R. Rach, Inversion of nonlinear stochastic operators. J. Math. Anal. Appl. **91**(1), 39–46 (1983)
18. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Methoc [ie Method]* (Kluwer Academic Publishers, Boston, 1994)
19. A. Wazwaz, Approximate solutions to boundary value problems of higher order by the modified decomposition method. Comput. Math. Appl. **40**(6–7), 679–691 (2000)
20. R. Singh, J. Kumar, G. Nelakanti, New approach for solving a class of doubly singular two-point boundary value problems using Adomian decomposition method. Adv. Numer. Anal. (2012). doi:10.1155/2012/541083
21. R. Singh, J. Kumar, The Adomian decomposition method with Green's function for solving nonlinear singular boundary value problems. J. Appl. Math. Comput. **44**(1–2), 397–416 (2014)
22. A. Wazwaz, R. Rach, Comparison of the Adomian decomposition method and the variational iteration method for solving the lane-emden equations of the first and second kinds. Kybernetes **40**(9/10), 1305–1318 (2011)
23. M. Hosseini, H. Nasabzadeh, On the convergence of Adomian decomposition method. Appl. Math. Comput. **182**(1), 536–543 (2006)
24. R.C. Rach, A new definition of the Adomian polynomials. Kybernetes **37**(7), 910–955 (2008)
25. E.H. Aly, A. Ebaid, R. Rach, Advances in the Adomian decomposition method for solving two-point nonlinear boundary value problems with neumann boundary conditions. Comput. Math. Appl. **63**(6), 1056–1065 (2012)
26. J.-S. Duan, Convenient analytic recurrence algorithms for the Adomian polynomials. Appl. Math. Comput. **217**(13), 6337–6348 (2011)